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FIRST PASSAGE PERCOLATION AND ESCAPE STRATEGIES

E. D. ANDJEL AND M.E. VARES

ABSTRACT. Consider first passage percolation on \mathbb{Z}^d with passage times given by i.i.d. random variables with common distribution F . Let $t_\pi(u, v)$ be the time from u to v for a path π and $t(u, v)$ the minimal time among all such paths from u to v . We ask whether or not there exist points $x, y \in \mathbb{Z}^d$ and a semi-infinite path $\pi = (y_0 = y, y_1, \dots)$ such that $t_\pi(y, y_{n+1}) < t(x, y_n)$ for all n . Necessary and sufficient conditions on F are given for this to occur.

1. INTRODUCTION

This work can be motivated by the following *game*: two individuals, called λ and σ , move on \mathbb{Z}^d , spending a random time $\tau(e)$ to cross each edge $e = \langle x, y \rangle$ from x to a nearest neighbor site y , or vice-versa. These passage times are assumed to be i.i.d. non-negative random variables (further assumptions will be made later). The two individuals start from distinct positions, that we denote by x_λ and x_σ respectively; λ would like to catch σ , who in turn wants to escape. A natural and simple question is: knowing the passage times, is σ able to devise a strategy that would be successful for his/her goal, independently of what λ does? In this case we shall say that σ has a *perfect* strategy. Thus, a perfect strategy is an infinite sequence of moves for σ in a way that (s)he will never be caught by λ regardless of what (s)he does. We are imagining that the individuals are always sitting on a vertex of \mathbb{Z}^d and think of an edge as being a door that remains closed unless someone *knocks* at it. To *open* a door corresponding to edge e an individual must be at one of the endpoints of e and knock it. Once this is done, the door will open after a time interval of length $\tau(e)$, the individual will cross that edge and the door will be closed immediately after. To prevent trivial situations it is natural to assume the common distribution to be *useful* in the sense of Definition 2.1 below; in particular $\{\tau(e) = 0\}$ does not percolate. As we shall see the situation changes significantly depending on the distribution of the passage times being supported on a compact set or not. When F has unbounded support, the probability that a perfect strategy for σ exists is zero, independently of the starting positions. On the other hand, in the bounded case and, if a clairvoyant σ can chose the initial position depending on x_λ and the τ variables, then a perfect strategy can be implemented with probability one. The result is precisely stated in Theorem 2.2 below. The proof in the bounded case is very simple. The unbounded case involves Proposition 3.1, which is indeed the main result of the paper.

We now outline the paper: in the next section we introduce the basic notation and definitions, and state two theorems. Theorem 2.2 has to do with the above question, and its easy part is

proved in the same section. Theorem 2.3 adds information to the comparison of first passage percolation models, treated by van den Berg and Kesten in [1] (in a more general case) in terms of time constants. In section 3 we state and prove Proposition 3.1 which is the main technical result of the paper, from which Theorem 2.3 and part (i) of Theorem 2.2 follow as immediate corollaries. We then conclude the proofs and discuss a related problem.

2. PRELIMINARIES AND RESULTS

Notation and definitions.

In this paper $\mathbb{E} = \mathbb{E}^d$ will denote the set of nearest neighbor (n.n.) edges in the cubic lattice \mathbb{Z}^d . The origin in \mathbb{Z}^d will be denoted by $\mathbf{0}$. For $x, y \in \mathbb{Z}^d$, $\|x - y\|$ will denote the ℓ_1 -distance, i.e. $\|x\| = \sum_{i=1}^d |x^i|$ for $x = (x^1, \dots, x^d) \in \mathbb{Z}^d$. A finite path $\pi = (e_1, \dots, e_k)$ is a sequence of adjacent edges (sharing a vertex), i.e. $e_i = \langle x_{i-1}, x_i \rangle$ for each $i = 1, \dots, k$. In this case we say that π goes from x_0 to x_k . For the context of this paper, it suffices to consider self-avoiding paths, i.e. when the $x_i, i = 0, \dots, k$ are all distinct, and we always assume this without further comment. Sometimes we identify a path with the sequence of its visited vertices, writing $\pi = (x_0, \dots, x_k)$.

The basic random object consists of a family $\{\tau(e) : e \in \mathbb{E}\}$ of i.i.d. non-negative random variables defined on a probability space (Ω, \mathcal{F}, P) , and where $\tau(e)$ represents the passage time at the edge e , interpreted as the time to traverse e . Their common distribution will be denoted by F . The passage time $t(\pi)$ of a given path $\pi = (e_1, \dots, e_k)$ is simply given by the sum of the variables $\tau(e_i)$ for $i = 1, \dots, k$. We say that a given path $\tilde{\pi}$ from x to y is optimal (from x to y) if its travel time is the shortest among all paths from x to y :

$$t(\tilde{\pi}) = \inf\{t(\pi) : \pi \text{ is a path from } x \text{ to } y\} =: t(x, y). \quad (2.1)$$

Any such optimal path is also called a *geodesic* (from x to y). An infinite path $\tilde{\pi} = (e_1, e_2, \dots)$ starting at x is said to be a semi-infinite geodesic if for any n the finite path (e_1, \dots, e_n) is a geodesic from x to its endpoint. It is easy to see that semi-infinite geodesics starting from any given point always exist. We also see easily that when F is continuous, there is a.s. a unique optimal path from x to y for any two distinct vertices x and y . Here, some assumptions on F will be needed, and as in [1] we set the following:

Definition 2.1. *A distribution F with support in $[0, +\infty)$ is called useful if the following holds:*

$$\begin{aligned} F(r) &< p_c \text{ when } r = 0, \\ F(r) &< \vec{p}_c \text{ when } r > 0, \end{aligned} \quad (2.2)$$

where p_c (\vec{p}_c) denotes the critical probability for the Bernoulli (oriented, resp.) bond percolation model on \mathbb{Z}^d , and r stands for the minimum of the support of F , hereby denoted by $\text{supp}(F)$.

Theorem 2.2. *Let F be useful in the sense of Definition 2.1.*

(i) If F has unbounded support, then for any x_λ, x_σ

$$P(\sigma \text{ has perfect strategy}) = 0. \quad (2.3)$$

(ii) Assume F to be supported in $[0, M]$ for some finite M . Let $\tilde{\pi}$ be a semi-infinite geodesic from x_λ . If the event

$$[M + t(x_\sigma, x) < t(x_\lambda, x) \text{ for some } x \in \tilde{\pi}] \quad (2.4)$$

occurs, then σ has a perfect strategy. In particular, given x_λ , with probability one there exist (infinitely many) random initial positions x_σ from where σ has a perfect strategy.

Proof. Part (i) will be proven in Section 3 as a corollary of our Proposition 3.1. We now prove only the easy part (ii). Indeed, under the situation described in (2.4), it follows at once that a perfect strategy for σ consists in taking any $x \in \tilde{\pi}$ for which $M + t(x_\sigma, x) < t(x_\lambda, x)$, moving to x by the geodesics from x_σ to x and then following the infinite branch of $\tilde{\pi}$ that starts in x . On the other hand, if F is useful it follows at once from the definitions that there exists $\delta > 0$ so that $F(\delta) < p_c$, which implies $t(x_\lambda, x) \rightarrow \infty$ as $x \rightarrow \infty$ along $\tilde{\pi}$, and the inequality in (2.4) becomes trivial for $x_\sigma \in \tilde{\pi}$ with $t(x_\lambda, x_\sigma) > M$.

Theorem 2.3. *Let F be a useful distribution on $[0, \infty)$ with unbounded support. Then, for each M positive there exists $\epsilon = \epsilon(M) > 0$ so that for all $n \geq 1$ and all x with $\|x\| = n$, we have*

$$P(\exists \pi \text{ geodesic from } 0 \text{ to } x \text{ such that } \tau(e) \leq M \text{ for all } e \in \pi) \leq e^{-\epsilon n}. \quad (2.5)$$

Remark. As is well known (see [4]) one may consider the time constant associated to the passage time distribution F , given by the deterministic limit (in probability):

$$\mu_F = \lim_{n \rightarrow \infty} \frac{1}{n} t(\mathbf{0}, (n, 0, \dots, 0)),$$

when F is the common distribution of the $\tau(e)$ variables, the limit being also a.s. and in L_1 under conditions on the tail of F , e.g. if F has finite mean. (The same result on asymptotic speed holds along any fixed direction, not only the coordinate axis. See [6, 2, 4, 5].) It is worth then to compare Theorem 2.3 with the following particular case of the more general results proven in [1], which imply that under the above conditions, the time constant for the distribution F truncated at M is strictly smaller than that for F : For any optimal path $\pi(n)$ from the origin to $(n, 0, \dots, 0)$, one has:

$$\liminf_{n \rightarrow \infty} \frac{1}{n} E \left(\sum_{e \in \pi(n)} \mathbf{1}_{\{\tau(e) > M\}} \right) > 0. \quad (2.6)$$

3. BASIC PROPOSITION. PROOFS.

Proposition 3.1. *Let F be a useful distribution on $[0, \infty)$ with unbounded support. For each $M > 0$ let*

$$\bar{t}_{(M)}(\mathbf{0}, x) = \inf\{t(\pi) : \pi \text{ is a path from } 0 \text{ to } x \text{ with } \tau(e) \leq M \text{ for all } e \text{ in } \pi\}, \quad (3.1)$$

with the understanding that $\inf \emptyset = +\infty$. Then, for each M positive there exists $\epsilon = \epsilon(M) > 0$ and $n_0 = n_0(M)$ so that for all $n \geq n_0$ and all x such that $\|x\| = n$, we have

$$P(\exists \text{ path } \tilde{\pi} \text{ from } \mathbf{0} \text{ to } x \text{ such that } M + t(\tilde{\pi}) < \bar{t}_{(M)}(\mathbf{0}, x)) \geq 1 - e^{-\epsilon n}. \quad (3.2)$$

The proof of Proposition 3.1 uses arguments from [1] that are recalled below. Before moving to this proof, let us observe that Theorem 2.3 is an immediate consequence of Proposition 3.1.

Notation. For $N \in \mathbb{N}$ and $l = (l^1, \dots, l^d) \in \mathbb{Z}^d$ consider the following partition of \mathbb{Z}^d by hypercubes, as in [1] called N -cubes.

$$S_l(N) = \{x \in \mathbb{Z}^d : Nl^i \leq x^i < Nl^i + N, \forall i\}. \quad (3.3)$$

The cubes are naturally indexed by l , and this indexing is also used to define the distance between two N -cubes. If $C \subset \mathbb{Z}^d$, we use $\mathcal{F}(C)$ to denote the σ -field generated by the variables $\tau(e)$ corresponding to edges e that have both endpoints in the set C .

Two other collections of boxes will be also useful in the proofs: for $N \in \mathbb{N}$, $l \in \mathbb{Z}^d$,

$$T_l(N) = \{x \in \mathbb{Z}^d : Nl^i - N \leq x^i \leq Nl^i + 2N, \forall i\}. \quad (3.4)$$

$$B_l^{\pm j}(N) = T_l(N) \cap T_{l \pm 2\mathbf{e}_j}(N), \quad j = 1, \dots, d,$$

where $\mathbf{e}_j, j = 1, \dots, d$ denote the canonical unitary vectors.

We first recall Lemma (5.2) from [1] (see also [3]) which follows from a Peierls argument:

Lemma 3.2. *If the cubes $S_l(N)$ are colored black or white in a random fashion which is (i) translation invariant; (ii) finite range (i.e. the color of $S_l(N)$ is $\mathcal{F}(\cup_{l'}(S(l', N) : \|l' - l\| \leq c_0))$ -measurable for a suitable constant c_0) and moreover, $P(S_0(N) \text{ is black}) \rightarrow 1$ as $N \rightarrow \infty$, then for all N sufficiently large we can find positive numbers $\epsilon = \epsilon(N)$ and $D = D(N)$ so that for each $u, v \in \mathbb{Z}^d$ the probability that each path from u to v visits at least $\epsilon\|u - v\|$ distinct black N -cubes is not smaller than $1 - e^{-D\|u - v\|}$.*

We must also recall Lemma (5.5) from [1], which says that if the distribution F of the time variables is useful, then there exist positive numbers $\delta = \delta(F)$ and $D_0 = D_0(F)$ such that

$$P(t(u, v) \leq (r + \delta)\|u - v\|) \leq e^{-D_0\|u - v\|}, \quad (3.5)$$

for all $u, v \in \mathbb{Z}^d$, where r is as in Definition 2.1.

Of course, for the proof of the proposition it suffices to consider $M > 0$ large and such that $P(\tau(e) \in (M, M+1]) > 0$. In the proof we shall also consider optimal paths for the passage times

$$\bar{\tau}(e) = \begin{cases} \tau(e) & \text{if } \tau(e) \leq M, \\ +\infty & \text{otherwise.} \end{cases}$$

Black boxes. We take $\delta = \delta(F)$ and $D_0 = D_0(F)$ so that (3.5) holds. We now say that the N -cube $S_l(N)$ is *black* if for any path π lying entirely in $T_l(N)$ with endpoints u, v such that $\|u - v\| \geq N/4$ we do have $t(\pi) \geq (r + \delta)\|u - v\|$. The N -cubes $S_l(N), S_{l'}(N)$ are said to be separated if $T_l(N) \cap T_{l'}(N) = \emptyset$.

From (3.5) we see that Lemma 3.2 applies. In particular, having fixed δ as above, for any N sufficiently large we can take $D = D(N, F)$ and $\epsilon = \epsilon(N, F)$ in a way that for all n large enough:

$$P(\exists \text{ path from } \mathbf{0} \text{ to } \Gamma_n \text{ that visits at most } \lceil \epsilon n \rceil \text{ separated black } N\text{-cubes}) \leq e^{-Dn}, \quad (3.6)$$

where $\Gamma_n = \{x \in \mathbb{Z}^d : \|x\| = n\}$ and $\lceil \cdot \rceil$ denotes the integer part. (Of course, changing D we may assume (3.6) holds for all n .)

We shall now work on the complement of the event on the l.h.s. of (3.6). For the proof of Proposition 3.1 we will try to improve over the optimal paths for $\bar{\tau}$ from 0 to some x in Γ_n by examining the probability of successful shortcuts in disjoint boxes $B_l^{\pm j}(N)$. The main point is the control of the conditional probability of a successful shortcut.

Definition 3.3. We say that a path π crosses the box $B_l^{\pm j}(N)$ if it crosses the box in the shortest direction and, except for its endpoints, is entirely contained in the interior of $B_l^{\pm j}(N)$.

Definition 3.4. 1) We say that a stretch $\pi_{[u,v]}$ of π is *shortcuttable* if it satisfies the following properties:

- (i) For some $i \in \{1, \dots, d\}$ $|u^i - v^i| = N$
- (ii) For all $z \in \pi_{[u,v]} \setminus \{u, v\}$ the i -th coordinate of z is strictly between u^i and v^i .
- (iii) For all $z \in \pi_{[u,v]}$ and all $j \in \{1, \dots, d\} \setminus \{i\}$ its j -th coordinate differs from u^j by at most $3N$, i.e. $|z^j - u^j| \leq 3N$
- (iv) For any pair x, y of points in $\pi_{[u,v]}$ such that $\|x - y\| \geq N/4$ we have $t(x, y) \geq (\delta + r)(\|x - y\|)$.

2) For $\rho > 0$ and small, we say that a path from the origin to Γ_n satisfies property $\mathcal{P}_n(\rho)$ if it contains at least $\lceil \rho n \rceil$ (integer part of ρn) shortcuttable stretches which lie at distance at least $7N$ of each other.

Remark 3.5. It follows from (3.6) that there are constants $\rho > 0$ and $D > 0$ such that for all n the probability that all paths from the origin to Γ_n satisfy condition $\mathcal{P}_n(\rho)$ is at least $1 - \exp(-Dn)$.

Properties (i)-(iii) in the above definition have the goal of characterizing the path as crossing a certain translate of the box $B_0^{\pm j}(N)$. The construction of a shortcut for such a piece is detailed below. Afterwards we discuss the problem of the conditional probabilities of having an effective reduction in the passage times.

Shortcuts

It is clear that if π is a path connecting a vertex x in $S_l(N)$ to $y \notin T_l(N)$, it must contain a path that crosses one of the $2d$ N -boxes $B_l^{\pm j}(N)$ in the sense just defined. Let π' be the first such crossings and call B the N -box that is crossed. Assuming π to be optimal for the $\bar{\tau}$ variables, we shall examine the possibility of a successful shortcut $\tilde{\pi}$ for π that uses an edge with passage time larger than M . This would be a path verifying the following conditions:

- $\tilde{\pi}$ and π are edge disjoint;
- the endpoints of $\tilde{\pi}$ coincide with those of a segment π'' of π ;
- $|\tilde{\pi}| \leq c_d N$ where the positive constant c_d depends only on the dimension;
- $\tilde{\pi}$ is contained in the same N -box B as π' .

We shall then say that a shortcut as above is *successful* if $M + t(\tilde{\pi}) < t(\pi'')$.

Let us first assume for notational simplicity that $d = 2$, $B = B_l^1(N)$, which we write as $B = [a, a + N] \times [b, b + 3N]$, and that π' crosses B from left to right. Writing $\pi = (x_0, \dots, x_s)$, let $v = x_j$ be the position in $\{a + N\} \times [b + 1, b + 3N - 1]$ where π first reaches the rightmost face of B after entering B and $u = x_i$ the position in $\{a\} \times [b + 1, b + 3N - 1]$ of the leftmost face of B last visited before getting to v , so that $i < j$ and π' is the segment of π that goes from u to v , which we denote as $\pi_{[u,v]}$. We choose $N = 4K$ for some $K \in \mathbb{N}$. We may define as well the vertex with lowest second coordinate and first coordinate in $[a + K, a + 3K]$ along $\pi_{[u,v]}$. If there are several such points, let us take e.g. the leftmost one, call it $z = (z^1, z^2)$. Let us assume for the moment that $z = \pi_k$ is on the leftmost half of B (i.e. $z^1 \leq a + N/2$); in this case we define $\tilde{\pi}$ by starting from z moving downwards one step to $z' = z - \mathbf{e}_2$ and then moving horizontally to the right for at most K steps or until we reach any point w in π , whatever comes earlier (note that we can have $w = z'$). In the first case, we then move upwards until reaching the first vertex w visited by π ; this just defined path from z to w is what we call $\tilde{\pi}$. Three cases have to be analyzed:

- (a) $w \in \pi_{[u,v]}$,
- (b) w is visited by π before u ,
- (c) w is visited by π after v .

In all of these three cases we define a new path substituting the stretch of π between w and z by $\tilde{\pi}$

In case (a) the substituted part is the stretch $\pi_{[z,w]}$ contained in $\pi_{[u,v]}$. In this case $\|z - w\| \geq K$.

In case (b) the substituted part is the portion of π going from w to z .

In case (c) the substituted part is the portion of π going from z to w .

It is easy to check that in all three cases the substituted part of π contains a stretch of $\pi_{[u,v]}$ connecting two points at distance at least $K = N/4$. If $\pi_{[u,v]}$ is shortcutable, then the time of the substituted part is at least $\|z - w\|r + K\delta$.

The extension to higher dimension is simple and we always have $\|z - w\| \leq 3Nd = 12Kd$. Assuming that the stretch $\pi_{[u,v]}$ is shortcuttable, a condition which guarantees a successful shortcut is $M + \sum_{e_i \in \tilde{\pi}} \tau(e_i) < \|z - w\|r + K\delta$. Note that the number of edges in $\tilde{\pi}$ is at most $\|z - w\| + 2$. For our application below (proof of Proposition 3.1), we shall impose for one of the edges, call it e_1 , that $\tau(e_1) \in (M, M + 1)$ and for the other edges we impose passage times in the interval $(r, r + \delta')$ with $\delta' = \delta/(24d)$. Hence, the shortcut is successful if

$$2M + 1 + (\|z - w\| + 1)(r + \delta') < \|z - w\|r + K\delta,$$

which is implied by

$$2M + 1 + r + \delta/(24d) + K\delta/2 < K\delta, \quad (3.7)$$

and this, in its turn, is satisfied for K large enough (depending on M , δ and r).

Proof of Proposition 3.1.

Let us fix δ as above and take $M > 0$ so that $P(\tau \in (M, M + 1]) > 0$. Moreover, we fix $N = 4K$ large so that Remark 3.5 holds and $2M + 1 + 2r + \delta/(24d) < K\delta/2$ as in the above construction. We may as well assume that the set on the right side of (3.1) is not empty, and let Π be a path where the minimum is attained. (In case of non-uniqueness, the argument will apply to any of the finitely many optimal paths, and any deterministic way to list them will do the job.)

We now define random variables $U_1, V_1, U_2, V_2, \dots$ taking values in $\mathbb{Z}^d \cup \{\infty\}$. On the event $\{\Pi = \pi\}$, U_1, V_1 are such that $\pi_{[U_1, V_1]}$ is the first shortcuttable stretch of π . If no such stretch exists then $U_1 = V_1 = \infty$; U_2, V_2 are such that $\pi_{[U_2, V_2]}$ is the first shortcuttable stretch of π after V_1 whose distance to $\pi_{[U_1, V_1]}$ is at least $7N$. In general, U_{i+1}, V_{i+1} is such that $\pi_{[U_{i+1}, V_{i+1}]}$ is the first shortcuttable stretch of Π after V_i whose distance to $\cup_{j=1}^i \Pi_{[U_j, V_j]}$ is at least $7N$, or $U_{i+1} = V_{i+1} = \infty$ if no such stretch exists.

For a given n let $q = q(n) = \lceil \rho n \rceil$. Then, partition the probability space in events as $A(\pi, x_1, y_1, \dots, x_q, y_q) = \{\Pi = \pi, U_i = x_i, V_i = y_i : 1 \leq i \leq q\}$ and the event $G = \{U_q = +\infty\}$. For each of the shortcuttable stretches of π there is a path $\tilde{\pi}_i$ as defined above, with z_i, w_i the corresponding vertices in that construction.

Call $e_{i,1}, \dots, e_{i,k_i}$ the edges of $\tilde{\pi}_i$ and call $e'_{i,1}, \dots, e'_{i,\ell_i}$ the edges which have one endpoint in $\tilde{\pi}_i \setminus \{w_i, z_i\}$ and whose other endpoint is not in $\tilde{\pi}_i$. We now define the event

$$F_i(\pi, x_1, y_1, \dots, x_q, y_q) = A(\pi, x_1, y_1, \dots, x_q, y_q) \cap \{\tau(e_{i,1}) \in (M, M + 1], \tau(e_{i,2}) < r + \delta', \dots, \tau(e_{i,k_i}) < r + \delta', \tau(e'_{i,1}) > M, \dots, \tau(e'_{i,\ell_i}) > M\}, \quad (3.8)$$

with $\delta' = \delta/(24d)$ as before. Notice that k_i, ℓ_i are uniformly (in i) bounded by a constant that depends only on K and d .

If the event $F_i(\pi, x_1, y_1, \dots, x_n, y_n)$ occurs then substituting a part of π as explained before we get a new path π'_i , and $M + t(\pi'_i) < t(\pi)$.

Thus, to conclude the proof it suffices to show that

$$P(\cap_{i=1}^q F_i^c(\pi, x_1, y_1, \dots, x_q, y_q) | A(\pi, x_1, y_1, \dots, x_q, y_q)) \leq (1 - \varepsilon)^q \quad (3.9)$$

for some $\varepsilon > 0$ (independent of q).

The proof of (3.9) will follow by a suitable application of the following simple lemma.

Lemma 3.6. *Let $\Omega = \mathbb{R}^\Lambda$, where Λ is a finite or countable set, endowed with the usual product Borel sigma-field $\sigma(\Lambda)$. Let Λ_1 be a (non-empty) finite proper subset of Λ and $\Lambda_2 = \Lambda \setminus \Lambda_1$. For $i = 1, 2$ let $\Omega_i = \mathbb{R}^{\Lambda_i}$, so that $\Omega = \Omega_1 \times \Omega_2$ and $\sigma(\Lambda) = \sigma(\Lambda_1) \times \sigma(\Lambda_2)$. Let μ_i be a Borel probability measure on $(\Omega_i, \sigma(\Lambda_i))$, $i = 1, 2$ and $\mu = \mu_1 \times \mu_2$ the product measure on $(\Omega, \sigma(\Lambda))$. If $A \in \sigma(\Lambda)$ and $\hat{B} \in \sigma(\Lambda_1)$ have the property that $x = (x_1, x_2) \in A$ and $y_1 \in \hat{B}$ imply $(y_1, x_2) \in A$, then*

$$\mu(B \cap A) \geq \mu(B)\mu(A),$$

where $B = \hat{B} \times \Omega_2$.

Proof. The hypothesis on A and \hat{B} can be written as

$$\mathbf{1}_{\hat{B}}(y_1)\mathbf{1}_A(x_1, x_2) \leq \mathbf{1}_{\hat{B}}(y_1)\mathbf{1}_A(y_1, x_2)$$

for all $x_1, y_1 \in \Omega_1$ and all $x_2 \in \Omega_2$. We compute the iterated integral $\mu_1(dy_1)\mu_1(dx_1)\mu_2(dx_2)$ on both sides. The left hand side yields, by Tonelli's theorem, $\mu(B)\mu(A)$. On the right hand side we have

$$\int_{\Omega_1} \mu_1(dy_1) \int_{\Omega_1} \mu_1(dx_1) \int_{\Omega_2} \mu_2(dx_2) \mathbf{1}_{\hat{B}}(y_1)\mathbf{1}_A(y_1, x_2)$$

which again by Tonelli's theorem can be rewritten as

$$\int_{\Omega} \mu(dy_1, dx_2) \mathbf{1}_{B \cap A}(y_1, x_2) \int_{\Omega_1} \mu_1(dx_1) = \mu(B \cap A)$$

proving the lemma. \square

To conclude the proof of Proposition 3.1, let

$$A = A(\pi, x_1, y_1, \dots, x_q, y_q) = \{\Pi = \pi, U_i = x_i, V_i = y_i : 1 \leq i \leq q\} \text{ and}$$

$$\hat{B}_i = \{\tau(e_{i,1}) \in (M, M+1], \tau(e_{i,2}) < r+\delta', \dots, \tau(e_{i,k_i}) < r+\delta', \tau(e'_{i,1}) > M, \dots, \tau(e'_{i,\ell_i}) > M\},$$

for $i = 1, \dots, q$, so that $P(\hat{B}_i) \geq \eta > 0$ for all i . A few instants of reflection show that the condition in the lemma is verified for the pair A and \hat{B}_1 : since Π is optimal for the $\bar{\tau}$ variables and has a finite time, it cannot cross any of the edges $e'_{1,1}, \dots, e'_{1,\ell_1}$; this prevents it from using the advantageous edges $e_{1,2}, \dots, e_{1,k_1}$, and therefore the modified configuration remains in A . Call F_i the event defined in (3.8). Since $F_1 = A \cap \hat{B}_1$, the lemma implies that $P(F_1^c \mid A) \leq 1 - P(\hat{B}_1)$. Analogously, we can again apply the lemma with A replaced by $A \cap \bigcap_{j=1}^{i-1} F_j^c$ for $i = 2, \dots, q$ and \hat{B}_i to conclude that conditional probability on the l.h.s. of (3.9) is bounded from above by $(1 - \eta)^{[qn]}$. \square

Proof of part (i) of Theorem 2.2.

It is clear that with probability one, no perfect strategy for σ can consist in remaining in a finite set for all times and so it must reach the set $\{x: \|x - x_\sigma\| = n\}$ for any n . It is obvious that on the event $\{t(x_\lambda, x_\sigma) < M\}$ any perfect strategy can only use edges with passage

time smaller than M and cannot include finite paths between two points whose passage time exceeds the minimal passage time between these points by more than M . Thus

$$P(\sigma \text{ has perfect strategy}, t(x_\lambda, x_\sigma) < M) \leq P(\exists x: \|x - x_\sigma\| = n, \bar{t}_{(M)}(x_\sigma, x) \leq M + t(x_\sigma, x))$$

Given $\eta > 0$, let M be such that $P(t(x_\lambda, x_\sigma) > M) \leq \eta$. Given such M we take n and ϵ so that such (3.2) holds and $c_d n^{d-1} e^{-\epsilon n} \leq \eta$, where $c_d n^{d-1}$ is an upper bound for the cardinality of $\{x: \|x\| = n\}$. We then have

$$P(\sigma \text{ has perfect strategy}) \leq 2\eta.$$

□

An open problem. Suppose that we have a family of independent Poisson processes $\{\mathcal{P}_e: e \in \mathbb{E}^d\}$ of parameter $c > 0$. Assume that the individuals λ and σ may now move from a vertex x to a n.n. vertex y at the jump times of \mathcal{P}_e where $e = \langle x, y \rangle$. In dimension one, as easily seen, the probability that σ has a perfect strategy is zero. What happens in higher dimensions?

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ENRIQUE D. ANDJEL, LATP URA 225/CNRS,
UNIVERSITÉ D’AIX-MARSEILLE,
39, RUE JOLIOT CURIE. 13453 MARSEILLE CEDEX 13, FRANCE
E-mail address: Enrique.Andjel@cmi.univ-mrs.fr

MARIA EULALIA VARES,
DME, INSTITUTO DE MATEMÁTICA, UNIVERSIDADE FEDERAL DO RIO DE JANEIRO,
AV. ATHOS DA SILVEIRA RAMOS 149, CEP 21941-909 - RIO DE JANEIRO, RJ, BRASIL
E-mail address: eulalia@im.ufrj.br